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Chern class identities from tadpole matching in type IIB and F-theory

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Abstract: In light of Sen's weak coupling limit of F-theory as a type IIB orientifold, the compatibility of the tadpole conditions leads to a non-trivial identity relating the Euler characteristics of an elliptically fibered Calabi-Yau fourfold and of certain related surfaces.

We present the physical argument leading to the identity, and a mathematical derivation of a Chern class identity which confirms it, after taking into account singularities of the relevant loci. This identity of Chern classes holds in arbitrary dimension, and for varieties that are not necessarily Calabi-Yau.

Singularities are essential in both the physics and the mathematics arguments: the tadpole relation may be interpreted as an identity involving stringy invariants of a singular hypersurface, and corrections for the presence of pinch-points. The mathematical discussion is streamlined by the use of Chern-Schwartz-MacPherson classes of singular varieties. We also show how the main identity may be obtained by applying 'Verdier specialization' to suitable constructible functions.

Keywords: F-Theory, D-branes, Differential and Algebraic Geometry

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Contents

1 Introduction

Orientifold compactifications of Type IIB string theory on a Calabi-Yau threefold in the presence of D3 and D7 branes can be geometrically described by F-theory compactified on a Calabi-Yau fourfold [43, 45]. Type IIB is defined in a ten-dimensional Minkowski space while F-theory requires two additional dimensions, which provide a geometric description of the axion-dilaton field of type IIB as the complex structure of an elliptic curve (a two-torus). Solutions of type IIB at weak coupling usually have a constant axion-dilaton field. F-theory provides a description of solutions with a variable axion-dilaton field by allowing the elliptic curve to be non-trivially fibered over a threefold. This construction provides a beautiful identification of S-duality in type IIB as the modular group of the elliptic curve.

Type IIB and F-theory are both severely constrained by 'tadpole conditions' which ensure that the total D-brane charges in a compact space vanish as required by Gauss's law. Tadpole conditions realize the physics wisdom according to which, in a compact space, the total charge should vanish since fluxes cannot escape to infinity. From a dynamical perspective, tadpole conditions are consistency requirements obtained from the local equations of motion and/or the Bianchi identities by integrating them over appropriate compact spaces.

The computation of tadpole conditions requires a detailed account of all contributions to the D-brane charges. This is closely related to anomaly cancellations since the presence of Chern-Simons terms in the D-brane action (needed for the cancellation of chiral and tensor anomalies) implies that a D-brane usually carries lower brane charges. In particular, in type IIB, a seven-brane has an induced D3 charge proportional to the Euler characteristic of the complex surface (a cycle of real dimension four in the Calabi-Yau threefold) on which it is wrapped. In F-theory, the induced D3 charge is proportional to the Euler characteristic of the Calabi-Yau fourfold. It follows that in the absence of other sources of D3 charge (like for example non-trivial fluxes), the consistency of the F-theory/type IIB tadpole relations leads to relations between the Euler characteristics of the F-theory fourfolds and the surfaces wrapped by the seven-branes.

The link between type IIB orientifolds and F-theory is clearly expressed in the case of $\mathbb{Z}_2$-orientifold symmetry by Sen's weak coupling limit of F-theory [43]. When the fourfold is realized as an elliptic fibration over a threefold, and Sen's weak coupling limit is used to produce the associated Calabi-Yau threefold, the relations can be recovered, as we show in this paper, at the price of dealing with singularities of the loci arising in the limit. In this paper we analyze one representative class of examples of this situation, presenting both the physical argument leading to the relation (§2), and a mathematical derivation of an identity of Chern classes which implies it (§4). In its form arising from physical considerations, the relation has the following shape. Let $\varphi : Y \to B$ be an E_8 elliptic fibration over a nonsingular threefold B, and assume that Y is a Calabi-Yau variety. Following Sen ([43]), we can associate with Y a Calabi-Yau threefold X, obtained as the double cover $\rho : X \to B$ ramified along a nonsingular surface O; at 'weak coupling limit', the discriminant of $Y \to B$ determines an orientifold supported on O, and a D7-brane supported on a surface D in X. Comparing the D3 tadpole condition as seen in F-theory and in type IIB leads to the (tentative) relation

$$2\,\chi(Y) \overset{?}{=} \chi(D) + 4\,\chi(O) \tag{1.1}$$

among Euler characteristics (cf. §2.6). However, the surface D is singular, and singular varieties admit several possible natural notions of 'Euler characteristic'; it is not *a priori* clear which one should be employed as $\chi(D)$ in (1.1). By contrast Y and O are both nonsingular, and $\chi(Y)$, $\chi(O)$ must refer to the usual topological Euler characteristic.

Analyzing this situation with mathematical tools, we can prove (Theorem 4.6) that in fact the relation (1.1) holds at the level of total homology Chern classes, provided that suitable correction terms are factored in to account for the singularities of D:

$$2\,\varphi_* c(Y) = \pi_* c(\overline{D}) + 4\,c(O) - \rho_* c(S) \tag{1.2}$$

implying

$$2\,\chi(Y) = \chi(\overline{D}) + 4\,\chi(O) - \chi(S) \quad . \tag{1.3}$$

Here, $\overline{D} \to D$ is a resolution of D (in fact, its normalization), $\pi : \overline{D} \to B$ is the composition $\overline{D} \to D \to B$, and S stands for the pinch locus of D. If $\dim B = 3$ (the case of physical interest), $\chi(S)$ simply counts the number of pinch-points of D. However (and surprisingly) (1.2) holds in arbitrary dimension, and independently of Calabi-Yau hypotheses. Thus, it appears that the scope of the tadpole conditions is actually substantially more general than the context in which they arise.

The term $\pi_* c(\overline{D})$ could be interpreted as the (push-forward to B of the) *stringy Chern class* of D, and the question remains of whether the corrected class $\pi_* c(\overline{D}) - \rho_* c(S)$, resp. its degree $\chi(\overline{D}) - \chi(S)$, are mathematically 'natural' notions. We take a stab at this question in §5, where a mechanism is proposed which appears to account precisely for the needed correction term, at least in the class of examples considered in this paper. We point out that the ingredients used to define the stringy Chern class (as in [3]) may in fact be employed to define other notions of Chern classes and Euler characteristics $\chi^{(m)}$ for singular varieties, depending on a parameter m. Each value of this parameter corresponds to a different candidate for 'relative canonical divisor' of the resolution map; for the examples considered in this paper, $m = 1$ corresponds to the notion leading to stringy invariants, while $m = 2$ corresponds to an alternative notion (leading to 'arc' invariants; the Ω-flavor considered in [3]). As we will see in §5, $\chi^{(m)}$ admits a well-defined limit as $m \to \infty$; and this Euler characteristic $\chi^{(\infty)}$ recovers precisely the relation (1.1) proposed by the string-theoretic considerations (Theorem 5.1). Therefore, this appears to be the natural notion in the context of this problem.

However, there is room for surprises, and it is not impossible that in more general situations the singularities modify the tapdole relation in a different ways than we have anticipated here. Although this will have no effect on the correctness of the mathematical result of this paper (the physics providing just an ansatz from the mathematical perspective) it would surely reshape the physics. To settle the issue, a complete physical derivation of the tapdole conditions taking into account the singularities would be appropriate. Tadpole conditions in string theory are usually obtained by a loop calculation or by using the inflow mechanism. Both roads have their shortcoming in presence of singularities.[1] It is therefore refreshing to know that the point of view presented in this paper is corroborated by an analysis of different physical aspects of the system ([7]). In particular it is shown in ([7]) that the choice of the Euler characteristic $\chi^{(\infty)}$ is compatible with a "deconstruction" description of the brane configuration in terms of a system of D9-anti-D9 branes with appropriate world-volume fluxes turn on.

This paper is aimed at both physicists and mathematicians; §2 is written with the former public in mind, and §4 with the latter. In order to enhance readability, these sections are essentially independent of each other, and the hurried reader of one sort may ignore the section more squarely written for the other. But the most interesting aspect of our results lies in the interplay between the two viewpoints.

We include in this introduction a few slightly more technical comments. The Chern

[1] A loop calculation requires a definition of the field theory in presence of singularities. This is usually possible when the singularities are very mild like for example if they are of the orbifold type. Performing an inflow calculation based on index theorems is also not free of additional assumptions since it will require extending the usual index theorem to singular varieties. Such an extension would depend on the choice of regularization of the singularities or in other words on the choice of definitions for the topological invariant of a singular variety. As we have seen, several non-equivalent choices are possible.

class identity we prove (Theorem 4.6) holds in arbitrary dimension, for varieties which are not necessarily Calabi-Yau, and generalizes in the simplest possible way the relation among Euler characteristics predicted by the tadpole condition presented in §2.6. In fact, Theorem 4.6 is established by lifting to this level of generality the elementary *Sethi-Vafa-Witten formula* (formula (2.12) in [44]) for the Euler characteristic of the fibration Y, and comparing it with an analogous formula obtained at Sen's weak coupling limit. We view the Sethi-Vafa-Witten formula as a general statement equating the Euler characteristic of the fibration with twice the Euler characteristic of a specific divisor G in the base B of the fibration (Proposition 4.2). The arguments in §4 are streamlined by using the calculus of *constructible functions,* which encodes the good properties of topological Euler characteristic and (by a result of R. MacPherson) of Chern classes. The relevant facts are recalled in §4. As a concrete example, we offer the following instance of the situation considered in this article.

Example 1.1. A degree 24 hypersurface with equation $y^2 = z^3+fz+g$ in weighted projective space $\mathbb{P}_{1,1,1,1,8,12}$ (with y, resp. z of degree 12, resp. 8, and f, g general polynomials in the other variables) determines a Calabi-Yau elliptic fibration $Y \to B$, with $B = \mathbb{P}^3$. Standard methods (for example, judicious use of the adjunction formula) easily yield $\chi(Y) = 23328$. Sen's weak coupling limit leads us to consider surfaces O, resp. $\underline{D}$ in $\mathbb{P}^3$ with equation $h = 0$, resp. $\eta^2 - 12h\chi = 0$, where h, η, χ are general polynomials of degrees 8, 16, 24 respectively. Again, adjunction yields immediately that $\chi(O) = 304$; as for $\underline{D}$, the presence of $8 \cdot 16 \cdot 24 = 3072$ nodes at the intersection S given by $h = \eta = \chi = 0$ has to be taken into account, and gives $\chi(\underline{D}) = 28864 - 3072 = 25792$.

The associated Calabi-Yau threefold X at weak coupling limit is the double cover of $\mathbb{P}^3$ branched over O; D is the inverse image of $\underline{D}$ in X. The orientifold and D7 brane are localized on O and D, respectively.

A simple local analysis shows that D is singular along a curve, with a set S of 3072 pinch-points corresponding to the set $\underline{S}$ of nodes on $\underline{D}$. It also shows that D may be resolved to a nonsingular surface $\overline{D}$ by blowing up the singular curve; the composition $\pi : \overline{D} \to \underline{D}$ is found to be 2-to-1 everywhere except over $\underline{S}$. In terms of constructible functions, this says that the push-forward of the constant function $1\!\!1_{\overline{D}}$ is

$$\pi_*(1\!\!1_{\overline{D}}) = 2 \cdot 1\!\!1_{\underline{D}} - 1\!\!1_{\underline{S}} \quad :$$

the function which equals 2 on $\underline{D} \smallsetminus \underline{S}$ and 1 on $\underline{S}$. It follows then immediately that

$$\chi(\overline{D}) = 2\chi(\underline{D}) - \chi(\underline{S}) = 2 \cdot 25792 - 3072 = 48512 \quad ;$$

what's more, the same relation must hold among the total Chern classes of these loci (cf. property (2) in §4).

We stress that more sophisticated intersection-theoretic tools are *not* needed in order to extract this information from the blow-up $\overline{D} \to D$. These simple considerations suffice to verify the tadpole relation with pinch-point correction in this example:

$$2 \cdot \chi(Y) = 2 \cdot 23328 = 46656 = 48512 + 4 \cdot (-304) - 3072 = \chi(\overline{D}) + 4\chi(O) - \chi(S) \quad .$$

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Note that ignoring singularities would have led us to an embarrassing tadpole mismatch between type IIB and F-theory at weak coupling[2]. The proof presented in §4 for the general case (at the level of Chern classes, in arbitrary dimension and without Calabi-Yau hypotheses) is no harder—modulo some intersection theory—than the proof sketched above for Example 1.1. Indeed, the key observation in the proof of Theorem 4.6 is precisely the same formula $\pi_*(1\!\!1_{\overline{D}}) = 2 \cdot 1\!\!1_{\underline{D}} - 1\!\!1_{\underline{S}}$ used above, which is just as easy to prove in general as in Example 1.1. In §4 we also offer an alternative, slightly more sophisticated viewpoint on such relations of constructible functions (by means of *Verdier specialization*, see Remark 4.5) as a possible venue for more general results. Intersection-theoretic invariants of singular varieties, including some computations involving blow-ups, also play a role in [9], in a comparison between the $E_8 \times E_8$ heterotic string and F-theory with G-fluxes.

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2 Physics

In this section we present in some detail the physical motivation for (1.1). Roughly speaking, it results from a direct comparison of the D3 brane tadpole condition in type IIB and in F-theory in the situation where no fluxes are turned on. We will introduce the necessary notions in a pedagogic way from (§2.1) to (§2.5); readers familiar with Sen's weak coupling limit of F-theory, tadpole conditions, and dualities, could consider jumping immediately to (§2.6), without great harm. In (§2.1), we give some basic notions on D-branes in type II string theories; in (§2.2) we review the (S-T-U-)dualities among type IIA, type IIB and M-theory. These dualities are important in understanding the dictionary between the physics and the geometry of F-theory and M-theory; they also provide an elegant derivation of the F-theory D3 brane tadpole condition. F-theory is introduced in (§2.3) and Sen's weak coupling limit is reviewed in (§2.4). Tadpoles and anomalies are discussed in (§2.5); the case of a $\mathbb{Z}_2$ orientifold of type IIB with O7-planes and D3 and D7 branes as well as the D3 tadpole condition in F-theory are worked out in detail.

[2]If we consider the case of several D7 branes, one can show that there is a unique configuration which satisfies the tadpole relation. Namely, two D7 branes wrapped around two smooth surfaces given respectively by polynomials of degree 28 and 4 in the Calabi-Yau three-fold. However, such a configuration is not generic in type IIB. Such non-generic configurations are discussed in [7], see also section 3 of the present paper.

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In (§2.6), we derive a general form of the relation (1.1) from §1 with several possible D7 branes and O7-planes. Relation (1.1), with only one D7 and one O7 branes, is the generic situation in Sen's weak coupling limit. However, since singularities are necessarily present in Sen's weak coupling limit, (1.1) must be modified to take them into account. Assuming that the final answer keeps the same shape, we give evidence that the modified formula would be of type of the relation (1.3). This will be confirmed in §4, where we analyze Sen's template situation in its natural mathematical setting, and prove formula (1.2) (which implies (1.3) for a larger class of varieties. For those who are mostly interested in the Calabi-Yau case, we note that (1.3) could be proved just by mimicking the treatment of example 1.1, using no more than the adjunction formula in the spirit of ([44]). While this computation is straightforward, the material in §4 provides a deeper understanding of the geometry of the situation.

2.1 Type II string theories and D-branes

There are five consistent ten-dimensional string theories. Here we will mostly be interested in type IIA and IIB string theories. Each of these two theories admit 32 supersymmetry generators organized into two ten-dimensional Majorana-Weyl spinors with opposite chirality in type IIA and the same chirality in type IIB. Both theories contain in their spectrum the following NS-NS (*Neveu-Schwarz*) fields: a graviton, an antisymmetric two-form which couples to the fundamental string, and a scalar field ϕ (called the *dilaton*) which controls the string coupling g_s in each of these theories, following the relation $g_s \sim e^{-\phi}$. They also contain RR (*Ramond-Ramond*) $(p+1)$-forms $C_{(p+1)}$ with $(p+1)$ odd in type IIA and even in type IIB. As a direct generalization of the charged particle in Maxwell theory, a $(p+1)$-form naturally couples to an object extended in p-spatial dimension. Indeed, as it evolves in spacetime, a p-brane draws a world-volume $W^{(p+1)}$ of spacetime dimension $(p+1)$ on which the $(p+1)$ potential $C_{(p+1)}$ can be evaluated as $\int_{W^{(p+1)}} C_{(p+1)}$. A p-brane charged under a $(p+1)$-form admits a magnetic dual which is a $(d-p-4)$-brane, where d is the spacetime dimension of the ambient space in which the brane lives. They are related by Hodge-conjugation $*F_{(p+2)} = F_{d-p-2}$ acting on their field strengths $F_{p+2} = \mathrm{d}C_{(p+1)}$.

The objects that carry the RR charges are not seen in perturbative string theory. It was realized by Polchinski ([37]) that p-branes are actually naturally present in string theory as loci on which open strings can end. For that reasons, they are usually called *Dirichlet p-branes* or (*Dp-branes*) since fixing the location of the ends of open strings is realized by imposing Dirichlet boundary conditions. D-branes are *half-BPS objects*, which means they preserve only half of the total amount supersymmetry. A string with its two ends on the same D-brane defines a U(1) gauge field (a U(1) bundle with a connection) on the world-volume of the brane. This is the *Chan-Paton bundle*. Note that since it is defined from an open string, the Chan-Paton gauge field defined on the world-volume of a D-brane is a NS-NS field.

Type IIA admits only Dp-branes with p even while p is odd in type IIB. More precisely, in type IIA we have a D0 brane (also called the *D-particle*) and a D2 brane, their magnetic duals are respectively the D6 and the D4 brane. There is also a D8 brane, which does not admits a magnetic dual. In type IIB there are a D(-1) (or *D-instanton*), D1 (or *D-*

string), D3, D5, D7 and D9 branes. The D7 and the D5 are the magnetic duals of the D-instanton and the D-string. The D9 does not admit a magnetic dual in ten dimensions. The fundamental string which is present both in type IIA and type IIB is usually called the *F-string,* and couples electrically to the NS-NS two-form. The magnetic dual of the F-string is a 5-brane (the *NS5 brane*) which is present both in type IIA and type IIB.

2.2 Dualities and M-theory

The five ten-dimensional string theories are related by a web of dualities which also include eleven-dimensional supergravity. The latter is the supersymmetric field theory of gravity with the highest possible spacetime dimension for the usual Minkowski signature. We shall review quickly some of these dualities in order to understand the origin of F-theory.

T-duality identifies the physics of type IIA compactified on a circle of radius r_A and type IIB compactified on a circle of radius r_B provided that these two radii are inverse of each other when measured in string units. A p-brane wrapped around the T-duality circle is T-dual to a $(p-1)$-brane not intersecting the T-duality circle and vice versa.

S-duality is a symmetry which relates the weak coupling regime of one theory to the strong coupling regime of another one. It opens a perturbative window in the strong coupling regime of a theory. Type IIB is its own S-dual. More precisely, the S-duality group in type IIB is a $\mathrm{SL}(2,\mathbb{Z})$ group of type IIB [27]. The S-dual theory of type IIA is an eleven-dimensional theory. The radius of the additional eleventh dimension grows as the coupling constant of type IIA increases. This eleven-dimensional theory is called M-theory and at low energy it is described by eleven-dimensional supergravity. The latter admits a three-form potential $A_{(3)}$ which can naturally couple to a membrane. This is the *M2-brane* in M-theory and its magnetic dual is 5-brane called the *M5-brane.*

U-duality relates type IIB and M-theory by using a combination of type IIA/IIB T-duality and type IIA/M-theory S-duality. More precisely, it provides a duality between type IIB string theory compactified on a circle S^1 and M-theory compactified on a torus $T^2 = S^1 \times S^1$, where the first circle is type IIA T-duality circle (of inverse radius than the type IIB radius) and the second one is the M-theory circle that controls the string coupling of type IIA. *Type IIB on a circle is dual to M-theory compactified on a torus whose area is shrinking to zero.* The modular group of this torus precisely corresponds to the $\mathrm{SL}(2,\mathbb{Z})$ S-duality group of type IIB theory. This geometrization of S-duality is the main interest of F-theory.

2.3 F-theory

We consider type IIB compactified on a Calabi-Yau threefold with D3 and D7-branes. All the branes are taken to be spacetime filling: they fill all the four-dimensional spacetime and are wrapped around a cycle of the Calabi-Yau threefold. A D3-brane will be point-like in the extra six dimensions and a 7-brane will wrap around a complex surface of the compact space while filling the four-dimensional spacetime. In type IIB we have two type of strings: the D-string has a RR charge while the F-string has a NS-NS charge. More generally, a (p,q)-string is the bound state of p F-strings and q D-strings [48] with p,q relatively prime

[41]; a (p,q)-brane is a brane on which (p,q)-strings can end [20, 24]. The usual D-brane is a $(1,0)$-brane.

The two real scalar-fields of type IIB are organized into a complex axion-dilaton field

$$\tau = C_{(0)} + \mathrm{i}e^{-\phi}, \tag{2.1}$$

where the axion is the RR-scalar $C_{(0)}$, and ϕ is the dilaton coming from the NS-NS sector; we recall that e^{ϕ} is the string coupling constant. The $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB acts on the axion-dilaton field as a modular transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{2.2}$$

The (p,q)-strings are the $\mathrm{SL}(2,\mathbb{Z})$ images of the fundamental string [20, 41]. The existence of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB forces us to contemplate the occurrence of (p,q)-branes for any relatively prime (p,q). However, this would require going above the usual weak coupling limit of type IIB, since the axion-dilaton field also changes under $\mathrm{SL}(2,\mathbb{Z})$ without preserving the scale of the string coupling constant: strong and weak coupling can be mapped into each other.

F-theory is a description of type IIB string theory in the presence of (p,q) 7-branes. These branes are non-perturbative objects which require a non-constant axion-dilation field. Since the axion-dilaton field τ is subject to modular transformations, Vafa [45] has proposed to describe it as the complex structure of an elliptic curve. *It is conjectured that F-theory on an elliptically fibered Calabi-Yau fourfold with a section and a base B is equivalent to type IIB on the base B with (p,q) 7-branes at the singular loci of the elliptic fibration.* The 7-branes are wrapped around divisors of the base B.

The modular group in F-theory is also the same as the modular group of the torus used to define U-duality between type IIB and M-theory. It follows that F-theory on an elliptically fibered Calabi-Yau is dual to M-theory on the same manifold in the limit where the fiber has a vanishing area. The three-form $A_{(3)}$ of M-theory reduces to the NS-NS and RR two-forms of type IIA under S-duality. These two-forms under T-duality give the NS-NS two-form of type IIB and the RR one-form of type IIB. We can then conclude that U-duality between type IIB and M-theory implies that an M2 brane wrapping the two torus defined by the T-duality circle of type IIA and the S-duality circle of M-theory will give rise to (p,q)-strings in F-theory.

2.4 The weak coupling limit of F-theory with E_8 fibrations

An E_8 elliptic fibration $\varphi : Y \to B$ has a Weierstrass normal equation

$$xy^2 - (z^3 + fzx^2 + gx^3) = 0, \tag{2.3}$$

written in a $\mathbb{P}^2$ bundle $\phi : \mathbb{P}(\mathcal{E}) \to B$. Here, f and g are respectively sections of powers $\mathcal{L}^4$, $\mathcal{L}^6$ of a line bundle $\mathcal{L}$ on the base B (cf. §4.1). The variety Y is a Calabi-Yau ($K_Y = 0$) if $c_1(\mathcal{L}) = c_1(TB)$. For every point of the base, the Weierstrass equation of the elliptic fibration defines an elliptic curve with (Klein's) modular function

$$j(q) = 4 \cdot \frac{(24f)^3}{\Delta}. \tag{2.4}$$

The function j is the generator of modular functions of weight one, and Δ is the discriminant of the elliptic curve:

$$\Delta = 4f^3 + 27g^2. \tag{2.5}$$

F-theory on the elliptically fibered Calabi-Yau fourfold Y is (conjecturally) equivalent to type IIB on the base B with $q = e^{2\pi i\tau}$. Since $\Im(\tau) = e^{-\phi} = \frac{1}{g_s}$ is the inverse of the string coupling constant, it follows that weak coupling ($g_s \ll 1$) corresponds to small q and therefore to large j since in the limit of small q we have $j \approx q^{-1}$. The 7-branes are located at points of the base manifold B where the elliptic fiber is singular ([43]); this is where $j(\tau)$ becomes infinite. These surfaces correspond to the vanishing locus of the discriminant Δ. *A priori,* the discriminant locus may have several components Δ_i which correspond to several $D7$-brane worldvolumes. Perturbative string theory on IIB background has two $\mathbb{Z}_2$ symmetries: the world sheet parity inversion Ω and the left-moving fermion number $(-1)^{F_L}$. Given a Calabi-Yau threefold X admitting an involution σ, we can mod out the spectrum of type IIB by the orientifold projection

$$\Omega \cdot (-1)^{F_L} \cdot \sigma^*, \tag{2.6}$$

where σ acts on the type IIB field via its pulback σ^*. In order to have only D3 and D7 branes, the involution σ is required to be holomorphic with the additional property $\sigma^*\Omega^{3,0} = -\Omega^{3,0}$, where $\Omega^{3,0}$ is the holomorphic three-form of the Calabi-Yau three-form X. Under the action of σ, the fixed locus is made of complex surfaces and/or isolated points. They correspond respectively to *orientifold 7-planes* (O7 planes) and O3 planes.

Following Sen ([43]) we set

$$\begin{aligned} f &= -3h^2 + c\eta, \\ g &= -2h^3 + ch\eta + c^2\chi, \end{aligned} \tag{2.7}$$

where c is a constant and h, η, χ are respectively general sections of line bundles $\mathcal{L}^2$, $\mathcal{L}^4$, $\mathcal{L}^6$ where $\mathcal{L}$ is the anticanonical bundle of B, as above. We recall that large j corresponds to large $\Im(\tau)$ and therefore to weak coupling. The limit $c \to 0$ is called the *weak coupling limit* since then $j(\tau)$ is large at every point of the base except in sectors where $|h|^2 \sim |c|$. Since in the weak coupling limit we have

$$\Delta \approx -9c^2h^2(\eta^2 + 12h\chi), \tag{2.8}$$

the zeroes of the dominant term of Δ are supported on $h = 0$. Sen shows ([43]) that $h = 0$ determines the locations of the $O7$-planes, while the surface $\underline{D}$ with equation $\eta^2 + 12h\chi = 0$ determines the locations of the $D7$-branes.

One can define a type IIB orientifold equivalent to the weak coupling limit of F-theory starting with a Calabi-Yau threefold X which is the double cover of the base B branched along $h = 0$ ([43]):

$$x_0^2 = h, \tag{2.9}$$

where x_0 is a section of the line bundle $\mathcal{L}$. The $\mathbb{Z}_2$-isometry that is gauged to describe the orientifold is $x_0 \to -x_0$. The fixed points under this symmetry correspond to the hypersurface $h = 0$. The variety X has vanishing first Chern class, and is therefore a Calabi-Yau manifold.

2.5 Tadpole conditions

It is natural to consider the surface $D \subset X$ obtained as inverse image of $\underline{D} \subset B$. In order to analyze D7-branes localized on D, we study the following general set-up.

We consider a D-brane wrapped around a cycle D of a Calabi-Yau manifold X. Open strings with both end points on the D-brane define the Chan-Paton bundle $E \to D$. The charge of the D-brane will depend on the embedding $f : D \hookrightarrow X$ and the topology of the Chan-Paton bundle E. This charge can be computed by an anomaly inflow argument ([32]). An anomaly is a violation of a symmetry of the Lagrangian by quantum effects; the anomaly of a gauge symmetry indicates an inconsistency of the theory and must vanish. The anomaly inflow mechanism [15, 16] consists of introducing an anomalous term in the Lagrangian to cancel the anomalous contribution of another term. The two terms will be anomalous when considered separately, but together they give an anomaly free theory.

Since string theory is anomaly free, self-consistency requires that for each possible anomaly there is a contribution in the Lagrangian that maintains the theory anomaly free. In other words, identifying a possible anomaly is an opportunity to discover a new sector of the Lagrangian of the theory. The new terms coming from anomaly inflow have been recovered by direct string theory calculations [17, 18, 34, 42]. In the case of D-brane configurations, an anomaly can be generated by massless chiral fermions or self-dual tensor fields located on the intersection of two branes. The cancellation of this chiral anomaly requires the presence of an anomalous term, usually called a Chern-Simons term or a Wess-Zumino term. The chiral anomaly is first computed using an index theorem. The Chern-Simons term is then deduced by a descent procedure. All these steps are purely algebraic and are by now well understood [8, 42]. Since the Chern-Simons term is linear in the RR potential, it gives a charge to the RR fields.

The Lagrangian for a RR field $C_{(p+1)}$ is of the type

$$\mathcal{L} = -\frac{1}{4} F \wedge *F + J \cdot C_{(p+1)}, \tag{2.10}$$

where $C_{(p+1)}$ is a $(p+1)$-form and $F = \mathrm{d}C_{(p+1)}$ is its field strength and $*F$ its Hodge dual; J is called the *current*. The equation of motion of C is

$$\mathrm{d}(*F) = J. \tag{2.11}$$

Using Gauss's law, the charge is the integral of the current. If we are in a compact space, the equations of motion tell us that the total charge should vanish:

$$Q_{\mathrm{Total}} = \int J = \int \mathrm{d}(*F) = 0. \tag{2.12}$$

This necessary condition is usually called a *tadpole condition.* See [38] for a pedagogic introduction.

2.5.1 Tadpoles in Type IIB

The Chern-Simons term for a D-brane wrapping a cycle D (admitting a Spin^c structure[3]) embedded in a Calabi-Yau threefold X ($f : D \hookrightarrow X$) with a Chan-Paton bundle E is given by[4] ([32]):

$$\int_X Q_D(f_* E) \wedge C = \int_D \mathrm{ch}(E') \wedge \sqrt{\frac{\hat{\mathrm{A}}(TD)}{\hat{\mathrm{A}}(ND)}} \wedge f^* C, \quad E' = E \otimes K_D^{-\frac{1}{2}}, \tag{2.13}$$

where $C = C_0 + C_2 + C_4 + C_6 + C_8 \in H^{\mathrm{even}}(X)$ is the total RR potential and f^*C its pullback to D; $\hat{A}$ is the total A-roof genus ([26]); TD is the tangent bundle to D and ND is its normal bundle; $\mathrm{ch}(E')$ is the total Chern character of twisted sheaf $E' = E \otimes K_D^{-\frac{1}{2}}$ where K_D is the canonical bundle of D. The appearance of the term $K_D^{-\frac{1}{2}}$ is related to the Freed-Witten anomaly [22, 28]. When the cycle wrapped by the D-brane is not a Spin but a Spin^c manifold, spinors are not section of the spin bundle Spin(D): such a bundle will suffer from a $\mathbb{Z}_2$ ambiguity. This ambiguity is cancelled by taking the tensor product with $K_D^{-\frac{1}{2}}$ since the latter admits the same ambiguity (K_D is always odd for a Spin^c manifold). This amounts to replacing E with the twisted bundle $E' = E \otimes K_D^{-\frac{1}{2}}$. Spinors are then sections not of Spin(D) but of the well-defined bundle Spin$(D) \otimes E'$. In presence of $\mathbb{Z}_2$-torsion in $H^2(D, \mathbb{Z})$, the canonical bundle K_D will admit more than one square root, and therefore there would be many possible Spin^c structure on D ([22]). Moreover, given a line bundle $\mathcal{M}$, one can also replace K_D by $K_D \otimes \mathcal{M}^2$ so that E' becomes $E' \otimes \mathcal{M}$. This reflects the freedom to choose different Chan-Paton bundles on a D-branes ([49]). In particular, when the manifold is Spin, one can choose $\mathcal{M}^2 = K_D$ so that the charge formula depends only on E. In this paper, when computing charges, we will always refer to the canonical Spin^c lift $E' = E \otimes K_D^{-\frac{1}{2}}$, even when D is Spin-manifold.

There is also a Chern-Simons term for an orientifold plane wrapped around a cycle O embedded in X as $i : O \hookrightarrow X$ (see for example [13, 42]):

$$\int_X Q_O \wedge C = -2^{p-4} \int_O \sqrt{\frac{\hat{\mathrm{L}}(TO/4)}{\hat{\mathrm{L}}(NO/4)}} \wedge i^* C, \tag{2.14}$$

where $\hat{\mathrm{L}}(S)$ is the Hirzebruch polynomial[5] of S and i^*C is the pullback of total RR potential C to O.

The Chern-Simons term of a Dp-brane or an Op-plane involves the total RR potential. It follows that a given p-brane has not only a p-brane charge but induces as well lower brane charges. The charge induced by the Chern-Simons term defines an element of $H^*(X, \mathbb{Z})$ called the *Mukai vector*. In the previous formulae Q_D and Q_O are respectively the Mukai vector of a D-brane and and O-plane wrapped respectively around a complex surface D

[3] See section 4.3 of ([49]) or §3 of ([10]).

[4] We consider the case where the NS-NS two-form vanishes and its field-strength has no discrete torsion.

[5] For any d, $\hat{\mathrm{L}}(dE)$ is defined as $\sum_k d^k\ \hat{\mathrm{L}}_k(E)$ where $\hat{\mathrm{L}}_k(E)$ is the term of degree k in the expansion $\hat{\mathrm{L}}(E) = \sum_j \hat{\mathrm{L}}_j(E)$.

and O. In type IIB with spacetime filling branes, the component of the Mukai vector of degree n represents the induced D$(9-n)$ brane charge.

For a D7 with a trivial Chan-Paton bundle E' and an O7 brane wrapped respectively around a complex surface D and O of a Calabi-Yau threefold X, we get the following Mukai vectors:

$$Q_D = [D] + \frac{\chi(D)}{24}\omega, \tag{2.15}$$

$$Q_O = -8[O] + \frac{\chi(O)}{6}\omega \tag{2.16}$$

where ω is the unit volume element of X. We have used $\int_S c_2(S) = \chi(S)$ and

$$\hat{\mathrm{A}}(S) = 1 - \frac{1}{24}(c_1^2 - 2c_2), \quad \hat{\mathrm{L}}(S) = 1 + \frac{1}{3}(c_1^2 - 2c_2), \quad c_i = c_i(S). \tag{2.17}$$

For a single D3 brane, we have

$$Q_{D3} = -\omega, \tag{2.18}$$

where ω is the volume density of the Calabi-Yau three-fold X (with $\int_X \omega = 1$) dual to a point; the conventional minus sign ensures that the D3 tadpole can be solved by introducing D3 branes of positive charge [7]. In a $\mathbb{Z}_2$ orientifold configuration without fluxes with N_{D3} D3-branes, D7-branes wrapping divisors D_i with trivial Chan-Paton vector bundle, and O7-planes wrapping around divisors O_j, the *tadpole condition* (cancellation of charges) reads:

$$\text{D7tadpole} : \sum_i [D_i] - 8\sum_j [O_j] = 0, \tag{2.19}$$

$$\text{D3tadpole} : N_{D3} = \frac{1}{2}\left(\sum_i \frac{\chi(D_i)}{24} + 4\sum_j \frac{\chi(O_j)}{24}\right), \tag{2.20}$$

where the indices i and j label respectively the D7-branes and the O7-planes. The factor of $\frac{1}{2}$ in the D3 tadpole takes into account the double counting of D3 charge in the cover space of a $\mathbb{Z}_2$ Calabi-Yau orientifold. Note also that the conventional minus sign in the charge of a single D3 brane ($Q_3 = -\omega$) ensures that the induced D3 charge coming from the curvature of the 7-branes is cancelled by N_{D3} D3-branes and not by N_{D3}-anti-D3 branes.

2.5.2 Tadpole in F-theory

F-theory compactified on an elliptically fibered Calabi-Yau four-fold Y admits a D3-tadpole condition which is obtained by a sequence of string dualities [44]. D3-branes are the only branes in type IIB invariant under $SL(2,\mathbb{Z})$. Therefore, in contrast to (p,q) 7-branes, D3 branes in F theory are essentially the same D3 branes seen in type IIB. In a sense, D3 branes play the same role as M2-branes in M-theory and fundamental strings in type IIA string theory. When an M2-brane is wrapped around the eleventh dimension used to relate M-theory and type IIA, it reduces to the F-string of type IIA, while an M2 brane that does not intersect the eleventh dimension of M-theory will give a D2-brane in type IIA. More

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precisely, the three-form of M-theory reduces to the NS-NS two-form that couples to the F-string of type IIA while the transverse part of $A_{(3)}$ reduces to the RR-three form $C_{(3)}$ that couples to the D2-brane. Moreover, under T-duality, a D3-brane wrapped around the T-duality circle will give a D2-brane in type IIB, under S-duality this D2 brane corresponds to an M2 brane transverse to the S-duality circle of M-theory. The determination of the F-theory tadpole can be simply deduced by reading the following sequence of dualities:

$$\begin{array}{ccccc} \text{IIA} & \xrightarrow{S-duality} & \text{M theory} & \xrightarrow{U-duality} & \text{Type IIB} \\ \text{F-string} & & \text{M2-brane} & & \text{D3 brane} \\ \int_{M_2\times Y} B_{(2)}\wedge Y_8 & & \int_{M_3\times Y} A_{(3)}\wedge Y_8 & & \int_{M_4\times Y} C_{(4)}\wedge Y_8 \end{array}$$

where Y_8 is a characteristic class for the four-fold Y such that $\int_Y Y_8 = \frac{\chi(Y)}{24}$ when Y is a Calabi-Yau[6] ([11]) and M_d represents the d-dimensional spacetime. Compactification of type IIA string theory on a Calabi-Yau four-fold Y to two dimensions leads to a tadpole term for the NS-NS two-form $B_{(2)}$ that couples to the fundamental string [46]. This tadpole is proportional to the Euler characteristic of Y. Since the corresponding type IIA interaction $\int B_{(2)}\wedge Y_8$ does not depend on the dilaton, it can be lifted to M-theory using S-duality, but the NS-NS two-form should be replaced by the three-form $A_{(3)}$ which couples to the M2 brane ([11, 21]). This new interaction $\int A_{(3)}\wedge Y_8$ can be seen as a quantum correction to the classical Chern-Simons term $\int A_{(3)}\wedge \mathrm{d}A_{(3)}\wedge \mathrm{d}A_{(3)}$ of eleven-dimensional supergravity. If we assume that there are no fluxes, the vanishing of the tadpole requires the presence of N_{M2} M2 branes, so that $N_{M2} = \frac{\chi(Y)}{24}$ (see chapter 10 of [12]). Finally, using U-duality between M-theory and F-theory, there is a similar tadpole for F-theory compactified on the Calabi-Yau four-fold Y, but this time for the four-form $C_{(4)}$ which couples to the D3 brane. The tadpole in type IIA is cancelled by the presence of NS-NS charge, in M theory it is cancelled by the presence of M2-branes charge while in F-theory compactified on an elliptically fibered Calabi-Yau four-fold Y, the tadpole is cancelled by D3 brane charge. If the latter is solely coming from N_{D3} D3 branes, it gives ([44]):

$$N_{D3} = \frac{\chi(Y)}{24}. \tag{2.21}$$

2.6 Matching F-theory and type IIB tadpole conditions

Consistency between type IIB and the F-theory D3 tadpole implies that

$$2\chi(Y) = \sum_i \chi(D_i) + 4\sum_j \chi(O_j), \tag{2.22}$$

by simply equating the expressions obtained for the number of D3 branes (N_{D3}) required in these two theories to satisfy the D3-tadpole condition. It is interesting to note that the two sides of this equality involve objects defined in different regimes. The elliptically fibered

[6] More precisely, we have $Y_8 = -\frac{1}{192}(c_1^4 - 4c_1^2c_2 + 8c_1c_3 - 8c_4)$, where here the Chern classes c_i are those of Y. When Y is a Calabi-Yau, $c_1 = 0$ and $Y_8 = \frac{c_4}{24}$.

Calabi-Yau four-fold Y is introduced to describe regimes in which the string coupling can be strong in presence of possible (p, q) 7-branes; on the other hand, the l.h.s. of the equality involves solely O-planes and $(1, 0)$ D7 branes which are only well-defined at weak coupling. This is well illustrated for example in ([43]), where an O7-plane is shown to correspond in strong coupling to a system of (p, q) 7-branes that coincide when the coupling becomes weak enough. The usual monodromy of the axion-dilaton field around such an O7-plane is reproduced as the total monodromy around the corresponding system of (p, q) 7-branes at strong coupling. It is therefore natural to consider the previous relation in a weak coupling limit of F-theory.

In Sen's weak coupling limit for E_8 elliptic fibrations, and for general choices of h, η, χ (and hence of f, g), we have a unique orientifold plane O and a unique D7 brane D in type IIB. Arguing as above, consistency between type IIB and F-theory D3 tadpole would give the equality presented in the introduction:

$$2\chi(Y) \stackrel{?}{=} \chi(D) + 4\chi(O) \quad . \tag{2.23}$$

However, equation (2.23) should be parsed carefully. The general arguments in §2.5 assume implicitly that all cycles under exam are *nonsingular* (for example, the expression of the Chern-Simons term assumes that the tangent bundle TD exists), while this is *not* the case for the surface D supporting the D7 brane in Sen's weak coupling limit. In the base B, the D7-brane is on the surface $\underline{D}$ defined by the equation $\eta^2 + 12h\chi = 0$. In the the Calabi-Yau threefold X, the inverse image D of $\underline{D}$ is defined by the equation

$$\eta^2 + 12x_0^2\chi = 0. \tag{2.24}$$

This surface is *singular* along the double curve $\eta = x_0 = 0$, and has pinch points (cf. [25], p. 617) at $\eta = x_0 = \chi = 0$.

There are in general several 'natural' definitions of Euler characteristic (or Chern class) of a singular variety, all giving the ordinary topological notions when applied to a nonsingular variety (see for example the appendix of [7]). The Euler characteristics of the Calabi-Yau fourfold Y and of the orientifold O are unambiguously defined, since these varieties are assumed to be nonsingular; but it is not clear how the term $\chi(D)$ should be interpreted in equation (2.23). Turning things around, we could consider equation (2.23) as giving a 'prediction' for $\chi(D)$. It is then natural to ask if this prediction matches a natural definition of Euler characteristic for a more general singular variety.

We will formulate some more concrete speculations along these lines in §5.

Another viewpoint on this situation is that, for more conventional Euler characteristics, the relation (2.23) should only be satisfied modulo a contribution from the singularities of D, which would vanish in the smooth case. The task amounts then to evaluating this correction term precisely.

This is accomplished in §4, with the result stated in the introduction: adopting the Euler characteristic of the *normalization* $\overline{D}$ of D as the natural notion of Euler characteristic for the singular surface D, we will find that the correction term needed in order to recover the tadpole relation (2.23) is *evaluated by the number of pinch-points of D*. Note that $\overline{D}$ is nonsingular, and that it can be identified as the blow-up of D along its singular locus.

Example 1.1 shows this phenomenon at work in a concrete instance, and the reader should have no difficulties adapting the proof given there to analyze the general case considered in §2.4, reaching the same conclusion. The physics underlying this particular example is analyzed in detail in ([7]).

3 Entr'act

Now that we have set the stage, we can address one doubt that may be lingering in the mind of the reader: is it truly necessary to invoke the presence of singularities in order to verify the tadpole condition? Might there not exist simpler configurations, consisting of D-branes supported on nonsingular surfaces, and simply satisfying the Euler characteristic constraints imposed by the tadpole condition?

In general (but with one notable exception, see below) this appears not to be the case: the relations are known not to hold when applied to examples where all loci are assumed to be smooth ([7]).

In F-theory, the seven-branes only wrap surfaces over which the elliptic fibration is singular. This restricts seriously the allowed configurations. For example, if we restrict ourself to $\mathbb{Z}_2$-orientifolds, in type IIB, the typical D7 configuration is composed of an O7-plane and a D7-brane; they wrap two complex surfaces that intersect along a curve. The D7 tadpole condition defines a linear relation among the homology cycles of these two surfaces.

In the case at hand, the O7 is supported on a smooth surface of class $c_1(\mathcal{L})$ in the Calabi-Yau threefold X; the D7 tadpole condition forces the D7 to be supported on a surface or a union of surfaces of total class $c_1(\mathcal{L}^8)$.

Now, if we assume that the D7 is supported on a single, smooth surface, then adjunction shows immediately that the tadpole matching condition of IIB and F-theory will simply *not* be satisfied. From a type IIB perspective, without any input from Sen's sharp description of the orientifold limit of F-theory, this would have been the typical configuration and would have not satisfied the tadpole matching condition of IIB and F-theory. However, this mismatch can be attributed to a naive identification of the typical configuration.

Assuming that the D7 is wrapped on a union of general smooth surfaces of varying classes equal to multiples of $c_1(\mathcal{L})$, one can verify (again using adjunction) that the matching is obtained for precisely one configuration: the O7-plane and two D7 branes wrapped around surfaces of classes $c_1(\mathcal{L})$, $c_1(\mathcal{L}^7)$, respectively. However, this configuration does not seem to be compatible with Sen's description of the orientifold limit of F-theory. Thus, singularities in the support of the D7 brane do appear to be a necessary feature of the situation, at least from the point of view of Sen's limit. As we have illustrated in Example 1.1, and as we are going to verify in general in §4, it is possible to satisfy the tadpole conditions within Sen's description, if we take seriously the presence of singularities.

The identity we will obtain by doing so will in fact realize the tadpole conditions at the level of Chern classes, and in arbitrary dimension. The configuration of two smooth surfaces mentioned above does not appear to generalize in the same fashion. This is further evidence that the configuration cannot be produced by a geometric construction analogous to Sen's description.

4 Mathematics

4.1 Discriminants of elliptic fibrations

We consider the following situation, extending the set-up of §2.4. Let B be a nonsingular compact complex algebraic variety of any dimension, and let $\varphi : Y \to B$ be an elliptic fibration, realized by a Weierstrass normal equation

$$y^2 x - (z^3 + f\, zx^2 + g\, x^3) = 0 \tag{4.1}$$

in a $\mathbb{P}^2$-bundle[7] $\phi : \mathbb{P}(\mathcal{E}) \to B$. Here (as in §2.4) f, resp. g are sections of powers $\mathcal{L}^4$, resp. $\mathcal{L}^6$ of a line bundle $\mathcal{L}$ on B. We can take $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}^3 \oplus \mathcal{L}^2$; the left-hand-side of (4.1) realizes Y as the zero-scheme of a section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(3) \otimes \phi^* \mathcal{L}^6$ in $\mathbb{P}(\mathcal{E})$.

We assume that the base loci of the linear systems $|\mathcal{L}^4|$, $|\mathcal{L}^6|$ are disjoint, and that f, g are general. We let $\Delta \subset B$ denote the discriminant hypersurface, given by

$$4\, f^3 + 27\, g^2 = 0 \quad ;$$

Δ is the zero-locus of a section of $\mathcal{L}^{12}$. The following is observed in [31], 1.5 (cf. [33], Proposition 2.1; and [35] for Weierstrass models):

Lemma 4.1. *With notation and assumptions as above:*

- *Y is nonsingular and φ is flat;*
- *for $p \notin \Delta$, the fiber $\varphi^{-1}(p)$ is a smooth elliptic curve;*
- *for $p \in \Delta$, $f(p) \neq 0$, the fiber $\varphi^{-1}(p)$ is a nodal cubic;*
- *for $p \in \Delta$, $f(p) = g(p) = 0$, the fiber $\varphi^{-1}(p)$ is a cuspidal cubic.*

We will denote by F, resp. G the hypersurfaces determined by f, g, and we will assume that F and G are nonsingular and intersect transversally. We also assume that Δ is nonsingular away from the codimension 2 locus

$$C : \quad f = g = 0 \quad ,$$

which is a nonsingular variety by the transversality hypothesis. All these assumptions are satisfied (by Bertini) if e.g., $\mathcal{L}$ is very ample.

As noted in §2.4, Y is a Calabi-Yau variety if and only if $\mathcal{L}$ is the anticanonical bundle of B ([31], 1.5 (i)). However, this hypothesis will not be needed for the main results in this section.

[7] We use the projective bundle of *lines* in $\mathcal{E}$.

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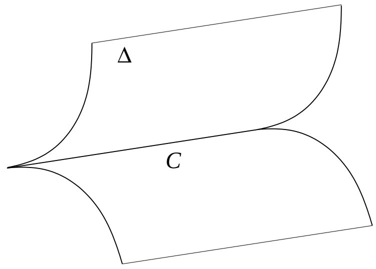

Figure 1. Local picture of the discriminant

4.2 Sethi-Vafa-Witten formula

Our first task is to extend the Sethi-Vafa-Witten formula (cf. §1) to the situation described above. We interpret this formula as a comparison between the total Chern class of the elliptic fibration Y and the total Chern class of the hypersurface G.

Proposition 4.2. *Let $\varphi : Y \to B$ be an elliptic fibration, as above. Then*

$$\varphi_* \, c(Y) = 2 \, \iota_* c(G) \quad ,$$

where $\iota : G \to B$ is the embedding of G.

Here and in the following, we denote by $c(X) = c(TX) \cap [X]$ the total 'homology' Chern class of the tangent bundle of X, *if X is a nonsingular variety.*

For the proof of Proposition 4.2, we rely on the calculus of constructible functions and Chern-Schwartz-MacPherson (CSM) classes on (possibly) singular varieties. We recall that a *constructible function* on a (complex, compact, possibly singular) variety X is a $\mathbb{Z}$-linear combination of characteristic functions of closed subvarieties of X. Constructible functions on X form an abelian group $F(X)$, which is covariantly functorial: if $\psi : X \to Y$ is a proper morphism of varieties, then ψ induces a homomorphism $\psi_* : F(X) \to F(Y)$ uniquely defined by the requirement that if $Z \subset X$ is a closed subvariety, then

$$\psi_*(1\!\!1_Z)(p) = \chi(\psi^{-1}(p) \cap Z) \quad ,$$

where $1\!\!1_Z$ is the characteristic function of Z, and χ denotes topological Euler characteristic. To each constructible function $\alpha \in F(X)$ we can associate a *CSM class* in the Chow group of X:

$$c_{\mathrm{SM}}(\alpha) \in A_*(X) \quad ,$$

such that

(1) if X is nonsingular, then $c_{\mathrm{SM}}(1\!\!1_X)$ equals the total Chern class of X:

$$c_{\mathrm{SM}}(1\!\!1_X) = c(X) \quad ;$$

(2) the assignment of CSM classes is functorial, in the sense that if $\psi : X \to Y$ is a proper morphism then $\forall \alpha \in F(X)$

$$\psi_*(c_{\mathrm{SM}}(\alpha)) = c_{\mathrm{SM}}(\psi_*(\alpha)) \quad .$$

In view of (1), one defines the *Chern-Schwartz-MacPherson class* of a (possibly singular) variety X as

$$c_{\mathrm{SM}}(X) := c_{\mathrm{SM}}(1\!\!1_X) \in A_*(X) \quad :$$

thus, $c_{\mathrm{SM}}(X) = c(X)$ if X is nonsingular, but $c_{\mathrm{SM}}(X)$ is defined for arbitrary varieties X. It easily follows from (2) that the degree of the zero-dimensional component of X is the topological Euler characteristic $\chi(X)$ of X.

In these definitions, the reader may replace the Chow group with ordinary integral homology. This topological setting was the context of the original result of MacPherson ([30]) and of earlier, independent work of Marie-Hélène Schwartz ([39, 40]); MacPherson's and Schwartz's very different definitions are known to lead to the same class ([6, 14]). For a rapid review of the definition of c_{SM} see [23], §19.1.7. An alternative is given in [4], Definition 3.2.

Although the statement of Proposition 4.2 only involves *nonsingular* varieties, its proof is considerably streamlined by using the notions we just recalled, which were introduced for the study of *singular* spaces.

Proof of Proposition 4.2. By Lemma 4.1 and the definition of push-forward of constructible functions recalled above, we have

$$\varphi_*(1\!\!1_Y) = 1\!\!1_\Delta + 1\!\!1_C \quad ;$$

by the properties of CSM classes recalled above,

$$\varphi_* c(Y) = \iota_{\Delta *} c_{\mathrm{SM}}(\Delta) + \iota_{C*} c(C) \quad ,$$

where ι_Δ, ι_C denote the corresponding embeddings (note that C is nonsingular).

The class $c_{\mathrm{SM}}(\Delta)$ may be computed by using Theorem I.4 from [2]:

$$c_{\mathrm{SM}}(\Delta) = c(TB) \cap \left(\frac{[\Delta]}{1+\Delta} + \frac{1}{1+\Delta} \left(s(\Delta_s, B)^\vee \otimes \mathcal{O}(\Delta) \right) \right) \quad , \tag{4.2}$$

where Δ_s denotes the *singularity subscheme* of Δ (defined locally by the ideal of partial derivatives of an equation for Δ), $s(\Delta_s, B)$ is its Segre class in B (cf. [23], Chapter 4), and $\mathcal{O}(\Delta)$ is the line bundle of which Δ is a section. By the assumptions detailed at the beginning of the section, Δ_s is supported on C. The differential of the (local) equation for Δ is $12f^2\, df + 54g\, dg$; the differentials df, dg are linearly independent in a neighborhood of C, and it follows that Δ_s has ideal

$$(f^2, g) \quad ,$$

that is, it is the complete intersection of G and the 'double' $2F$ of F. Since the complete intersection of F and G is C, we conclude

$$s(\Delta_s, B) = \frac{2[C]}{(1+2F)(1+G)} \quad .$$

Applying (4.2) we get:[8]

$$c_{\rm SM}(\Delta) = c(TB)\cap\left(\frac{[\Delta]}{1+\Delta}+\frac{1}{1+\Delta}\frac{2[C]}{(1-2F)(1-G)}\otimes\mathcal{O}(\Delta)\right) \tag{4.3}$$

$$= c(TB)\cap\left(\frac{[\Delta]}{1+\Delta}+\frac{1}{1+\Delta}\frac{2[C]}{(1+\Delta-2F)(1+\Delta-G)}\right) \tag{4.4}$$

Since $\Delta = 3F = 2G$ as divisor classes, and $[C] = F\cdot[G]$, this shows

$$c_{\rm SM}(\Delta) = c(TB)\cap\left(\frac{[\Delta]}{1+\Delta}+\frac{1}{1+\Delta}\frac{2[C]}{(1+F)(1+G)}\right) = c(TB)\cap\frac{(2+F)\cdot[G]}{(1+F)(1+G)} \quad .$$

Using this, and again the fact that C is the complete intersection of F and G,

$$\varphi_* c(Y) = c(TB)\cap\left(\frac{(2+F)\cdot[G]}{(1+F)(1+G)}+\frac{[F][G]}{(1+F)(1+G)}\right) \tag{4.5}$$

$$= c(TB)\cap\frac{2\,[G]}{1+G} = 2\,\iota_* c(G) \quad , \tag{4.6}$$

giving the statement. □

Since the Euler characteristic agrees with the degree of the top Chern class, we obtain the following statement (in terms of the class of the line bundle $\mathcal{L}$ introduced at the beginning of the section):

Corollary 4.3. *Let Y be an elliptic fibration on a nonsingular compact variety B, as above. Let $\ell = c_1(\mathcal{L})$, and $c_i = c_i(TB)$, and let $b = \dim B$. Then*

$$\chi(Y) = 12\,\ell\left(c_{b-1} - 6\,\ell\, c_{b-2} + 6^2\,\ell^2\, c_{b-3} + \cdots + (-6)^{b-1}\,\ell^{b-1}\right) \quad .$$

Proof. Since G has class 6ℓ,

$$\iota_* c(G) = c(TB)\,\frac{6\ell}{1+6\ell}\cap[B] \quad .$$

Applying Proposition 4.2, and reading off the term of dimension 0, yields the statement. □

Proposition 4.2 and Corollary 4.3 hold regardless of any Calabi-Yau hypothesis. As observed above, Y is a Calabi-Yau variety if and only if $\ell = c_1$; in this case, Corollary 4.3 reproduces Proposition 2 in §8 of [29]. Some values for $\chi(Y)$ are given in the table 1. The third line in the table (that is, Corollary 4.3 for $\dim B = 3$ and $\ell = c_1$) reproduces formula (2.12) in [44]. In this sense, Proposition 4.2 should be viewed as a generalization of the Sethi-Vafa-Witten formula—holding in arbitrary dimension, for E_8 elliptic fibrations that are not necessarily Calabi-Yau varieties, and at the level of total Chern classes.

[8]Here we use the simple calculus of the operations $\otimes$, $^\vee$, see §2 in [1]. The expression $\frac{2[C]}{(1+2F)(1+G)}$ is viewed as $c(\mathcal{A})^{-1}\cap a$, where $\mathcal{A}$ is a vector bundle with roots $2F$, G, and a is the homology class $2[C]$. Now, for all vector bundles $\mathcal{A}$:

$$(c(\mathcal{A})^{-1}\cap a)^\vee = c(\mathcal{A}^\vee)^{-1}\cap a^\vee \quad ,$$

and for all line bundles $\mathcal{M}$

$$(c(\mathcal{A})^{-1}\cap a)\otimes\mathcal{M} = c(\mathcal{M})^{\mathrm{rk}\mathcal{A}} c(\mathcal{A}\otimes\mathcal{M})^{-1}\cap(a\otimes\mathcal{M}) \quad .$$

The classes $a^\vee$ and $a\otimes\mathcal{M}$ are both linear in a. For a pure-dimensional a, they equal $(-1)^{\mathrm{codim}\,a}a$, $c(\mathcal{M})^{-\mathrm{codim}\,a}\cap a$, respectively.

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dim B	$\chi(Y)$	for Y a Calabi-Yau ($\ell = c_1$):
1	$12\,\ell$	$12\,c_1$
2	$12\,\ell(c_1 - 6\,\ell)$	$12\,c_1(-5\,c_1)$
3	$12\,\ell(c_2 - 6\,\ell c_1 + 6^2\ell^2)$	$12\,c_1(30\,c_1^2 + c_2)$
4	$12\,\ell(c_3 - 6\,\ell c_2 + 6^2\,\ell^2 c_1 - 6^3\ell^3)$	$12\,c_1(-180\,c_1^3 - 6\,c_1 c_2 + c_3)$
5	$12\,\ell(c_4 - 6\,\ell c_3 + 6^2\,\ell^2 c_2 - 6^3\,\ell^3 c_1 + 6^4\ell^4)$	$12\,c_1(1080\,c_1^4 + 36\,c_1^2 c_2 - 6\,c_1 c_3 + c_4)$

Table 1. Euler characteristic of E_8 elliptic fibrations

4.3 Tadpole relation

Our next goal is to analyze the tadpole relation of §2.6 in the context of the situation presented in §4.1: that is, to obtain a precise relation for the Euler characteristics, holding in arbitrary dimension, and bypassing the Calabi-Yau hypothesis. We first obtain a Chern class relation involving the discriminant determined in §4.1 and the discriminant at weak coupling limit; then we interpret the result in terms akin to those presented in §2.6.

Recall that the weak coupling limit is obtained by viewing the defining equation (4.1)

$$y^2x - (z^3 + f\,zx^2 + g\,x^3) = 0$$

as a perturbation of the degenerate fibration

$$y^2x - (z^3 + (-3h^2)\,zx^2 + (-2h^3)\,x^3) = 0 \quad ,$$

where h is a general section of $\mathcal{L}^2$. More precisely, we let

$$\begin{cases} f = -3h^2 + c\,\eta \\ g = -2h^3 + c\,h\eta + c^2\,\chi \end{cases}$$

where c is a scalar, and η, resp. χ are general sections of $\mathcal{L}^4$, resp. $\mathcal{L}^6$. For general c, we are in the situation presented in §4.1; the weak coupling limit is obtained by letting $c \to 0$. The resulting family of discriminants has flat limit

$$h^2(\eta^2 + 12h\chi)$$

for $c = 0$. The corresponding hypersurface $\underline{\Delta}$ is the union of a (double) nonsingular component O with equation $h = 0$, and the singular hypersurface $\underline{D}$ given by $\eta^2 + 12h\chi = 0$. Under our standing generality assumptions, the only singularities of $\underline{D}$ are along the (transversal) intersection $h = \eta = \chi = 0$, a nonsingular subvariety $\underline{S}$ of codimension 3 in B. In fact, as $d(\eta^2 + 12h\chi) = 2\eta\,d\eta + 12h\,d\chi + 12\chi\,dh$ and $d\eta, d\chi, dh$ are independent along S, it follows that the singularity subscheme $\underline{D}_s$ of $\underline{D}$ coincides with $\underline{S}$.

We now consider the problem of expressing $\varphi_*(c(Y))$ in terms of this limiting discriminant, analogously to Proposition 4.2. The following should be viewed as a 'limiting Sethi-Vafa-Witten formula':

Lemma 4.4. *With notation as above,*

$$\varphi_* c(Y) = c_{\rm SM}(1\!\!1_{\underline{D}} + 2\, 1\!\!1_O - 1\!\!1_{\underline{S}}) \quad .$$

Proof. Since O and $\underline{S}$ are nonsingular,

$$c_{\rm SM}(O) = c(TB) \cap \frac{[O]}{1+O} \quad , \quad c_{\rm SM}(\underline{S}) = c(TB) \cap \frac{[\underline{S}]}{(1+O)(1+F)(1+G)}$$

by the normalization property (1) of CSM classes, and noting that η, resp. χ are sections of $\mathcal{L}^4 \cong \mathcal{O}(F)$, resp. $\mathcal{L}^6 \cong \mathcal{O}(G)$.

The class $c_{\rm SM}(\underline{D})$ is evaluated by again applying Theorem I.4 from [2]:

$$c_{\rm SM}(\underline{D}) = c(TB) \cap \left(\frac{[\underline{D}]}{1+\underline{D}} + \frac{1}{1+\underline{D}} \left(s(\underline{D}_s, B)^\vee \otimes \mathcal{O}(\underline{D}) \right) \right) \quad .$$

We have already observed that this scheme is in fact $\underline{S}$, hence

$$s(\underline{D}_s, B) = s(\underline{S}, B) = \frac{[\underline{S}]}{(1+O)(1+F)(1+G)} \quad .$$

Applying the 'calculus' recalled in footnote 8:

$$c_{\rm SM}(\underline{D}) = c(TB) \cap \left(\frac{[\underline{D}]}{1+\underline{D}} - \frac{1}{1+\underline{D}} \frac{[\underline{S}]}{(1+O)(1+F)(1+G)} \right) \quad .$$

Using that $[\underline{D}] = 2[F] = [O] + [G]$, $[G] = [O] + [F]$, and $[\underline{S}] = [O][F][G]$, it follows that

$$c_{\rm SM}(\underline{D}) + 2 c_{\rm SM}(O) - c_{\rm SM}(\underline{S}) = c(TB) \cap \frac{2[G]}{1+G} = 2\, \iota_* c(G) \quad .$$

The statement follows then immediately from Proposition 4.2. □

Remark 4.5 (Verdier specialization). According to Lemma 4.4, the two constructible functions

$$1\!\!1_\Delta + 1\!\!1_C \qquad \text{and} \qquad 1\!\!1_{\underline{D}} + 2 1\!\!1_O - 1\!\!1_{\underline{S}}$$

have the same CSM class in $A_*(B)$. It is natural to ask for 'systematic' ways to produce such identities. One possibility consists of studying the *Verdier specialization* σ_X of the constant function 1 from the total space of a smoothing family for a hypersurface X. For an efficient summary of this notion, introduced in [47], we recommend §5 of [36]; the function σ_X is essentially defined by taking Euler characteristics of nearby fibers. In the case at hand, the reader can verify that

$$\sigma_\Delta = 1\!\!1_\Delta - 2\, 1\!\!1_C + 2\, 1\!\!1_{C'}$$

and

$$\sigma_{\underline{\Delta}} = 1\!\!1_{\underline{D}} + 2 1\!\!1_O - 6 1\!\!1_Q + 3 1\!\!1_{\underline{S}} - 1\!\!1_{O'} + 5 1\!\!1_{Q'} - 3 1\!\!1_{\underline{S}'} \quad ,$$

where $Q = O \cap \underline{D}$, and primed letters denote the intersection of the corresponding locus with a general element of the linear system of the hypersurface (use [36], Proposition 5.1).

Now, it is easy to see that the CSM class of the specialization function σ_X only depends on the divisor class of X (in fact, it reproduces the Chern class of the virtual tangent bundle of X); hence

$$c_{\rm SM}(\sigma_\Delta) = c_{\rm SM}(\sigma_{\underline{\Delta}}) \quad .$$

Further, one can verify directly that

$$c_{\rm SM}(6 1\!\!1_Q - 3 1\!\!1_C + 1\!\!1_{O'} - 4 1\!\!1_{\underline{S}} - 5 1\!\!1_{Q'} + 2 1\!\!1_{C'} + 3 1\!\!1_{\underline{S}'}) = 0 \quad :$$

this is straightforward since every locus appearing on the l.h.s. is a nonsingular complete intersection. Combining these identities yields precisely that

$$c_{\rm SM}(1\!\!1_\Delta + 1\!\!1_C) = c_{\rm SM}(1\!\!1_{\underline{D}} + 2 1\!\!1_O - 1\!\!1_{\underline{S}}) \quad .$$

This gives an alternative argument for Lemma 4.4, bypassing the use of [2] and shedding some light on the reason why such an identity should hold in the first place.

Next, we consider (as in §2.4) the double cover $\rho : X \to B$ ramified along the smooth hypersurface O. Note that X has vanishing canonical class when $[O] = 2\, c_1(TB) \cap [B]$; it follows that X is a Calabi-Yau if Y is a Calabi-Yau. However, again we point out that this hypothesis is not necessary for the considerations in this section.

We denote by D the inverse image of $\underline{D}$ in X. The local analysis summarized in §2.6 goes through unchanged in the situation considered in this section. Explicitly: X may be realized (as in §2.6) by putting $h = x_0^2$; we may use $\eta, \chi, x_0, x_1, \dots, x_r$ as local coordinates in X, and D is then defined (locally) by $\eta^2 + 12 x_0^2 \chi = 0$. This hypersurface is singular along $\eta = x_0 = 0$, corresponding to the inverse image of $Q = O \cap \underline{D}$, and 'pinched' along $\eta = \chi = x_0 = 0$, that is, the inverse image $S = \rho^{-1}(\underline{S})$. Note that ρ restricts to an isomorphism $S \to \underline{S}$. We consider the resolution $\overline{D}$ of D obtained by blowing up $\eta = x_0 = 0$: locally, the blow-up of X is covered by two charts, and we may choose local coordinates in one of these charts as follows:

$$\tilde\eta, \chi, \tilde x_0, x_1, \dots, x_r$$

so that the blow-up map is given by

$$(\tilde\eta, \chi, \tilde x_0, x_1, \dots, x_r) \mapsto (\tilde\eta \tilde x_0, \chi, \tilde x_0, x_1, \dots, x_r) \quad .$$

In these coordinates, the inverse image of D is given by the equation $\tilde x_0^2(\tilde\eta^2 + 12\chi) = 0$; therefore the blow-up $\overline{D}$ of D has equation

$$\tilde\eta^2 + 12\chi = 0 \quad ,$$

and in particular it is nonsingular in this chart. The situation on the other local chart of the blow-up can be analyzed similarly, with the same conclusion: blowing up the singular locus of D resolves its singularities.[9]

[9] If D is a surface, as in §2.6, this is of course the standard resolution of the Whitney umbrella obtained by blowing up along the singular curve.

Note that the map $\overline{D} \to D \to \underline{D}$ is generically 2-to-1, and 1-to-1 precisely over $\underline{S}$. We denote by

$$\pi : \overline{D} \to B$$

the composition $\overline{D} \twoheadrightarrow D \twoheadrightarrow \underline{D} \hookrightarrow B$.

Theorem 4.6. *With notation as above,*

$$2\,\varphi_* c(Y) = \pi_* c(\overline{D}) + 4\,c(O) - \rho_* c(S)$$

in $A_*(B)$.

Proof. The map π is 2-to-1 onto the complement of $\underline{S}$ in $\underline{D}$, and 1-to-1 over the singular locus $\underline{S}$ of $\underline{D}$. Therefore, by definition of push-forward of constructible functions,

$$\pi_*(1\!\!1_{\overline{D}}) = 2\,1\!\!1_{\underline{D}} - 1\!\!1_{\underline{S}} \quad .$$

It follows that

$$\pi_*(1\!\!1_{\overline{D}}) + 4\,1\!\!1_O - \rho_*(1\!\!1_S) = 2\,1\!\!1_{\underline{D}} + 4\,1\!\!1_O - 2\,1\!\!1_{\underline{S}} \quad ,$$

and hence, by Lemma 4.4,

$$c_{\rm SM}(\pi_*(1\!\!1_{\overline{D}}) + 4\,1\!\!1_O - \rho_* 1\!\!1_S) = 2\,c_{\rm SM}(1\!\!1_{\underline{D}} + 2\,1\!\!1_O - 1\!\!1_{\underline{S}}) = 2\,\varphi_* c(Y) \quad .$$

The formula given in the statement follows from this by the properties (1), (2) of CSM classes recalled in §4.2. □

Considering only the term of dimension 0 in Theorem 4.6 gives

Corollary 4.7.

$$2\,\chi(Y) = \chi(\overline{D}) + 4\,\chi(O) - \chi(S) \quad .$$

If $\dim B = 3$, so that S consists of a discrete set of points, then $\chi(S)$ simply equals the number of pinch-points of the surface D. This relation is the corrected version of the tadpole relation (2.23) promised in §2.6. It is generalized to arbitrary dimension, and emancipated from the Calabi-Yau hypothesis.

5 Speculation

The Euler characteristic $\chi(\overline{D})$ appearing in Corollary 4.7 could be interpreted as the *stringy Euler characteristic* of D, although D is not normal: the blow-up $\overline{D} \to D$ is 'crepant' in the sense that its differential is regular in codimension 1. By the same token, the push forward of $c(\overline{D})$ should be interpreted as the *stringy Chern class* of the singular hypersurface $D \subset B$, cf. [3] and [19].

In fact, the machinery of [3] produces other 'natural' notions of Chern class (and, in particular, of Euler characteristic) for singular varieties, depending on how the relative canonical divisor of a resolution is handled. For a review of these notions, the reader is addressed to [5]. Briefly: if $\nu : \overline{Z} \to Z$ is a resolution of a singular variety Z, the 'celestial integral'

$$\int_{\overline{Z}} \mathbb{1}(K_\nu)\, d\mathfrak{c}_{\overline{Z}}$$

determines a class in $(A_*Z)_{\mathbb{Q}}$, which may be taken as defining a 'total Chern class' for Z. Here, K_ν denotes the chosen notion of relative canonical divisor of ν, which in turns depends on how ω_Z is defined:

- Taking $\wedge^{\dim Z}\Omega^1_Z$ leads to the *arc Chern class*[10] of Z, $c_{\mathrm{arc}}(Z)$;
- Taking the double-dual of $\wedge^{\dim Z}\Omega^1_Z$ leads to the *stringy Chern class* $c_{\mathrm{str}}(Z)$. This notion was independently defined and studied in [19].

The degree of the Chern class recovers the corresponding notion of Euler characteristic. If Z is nonsingular to begin with, all these notions coincide and simply reproduce the usual Chern class and (topological) Euler characteristic of Z.

For the situation considered in §4, we have the resolution

$$\nu : \overline{D} \to D$$

obtained by blowing up a codimension 1 locus in D. A computation in local coordinates shows that $\Omega^{\dim D}_{\overline{D}|D}$ is supported on the inverse image $\overline{S} = \nu^{-1}(S)$ of the pinch locus of D; this is a codimension 2 locus, and it follows that the relative canonical divisor in the 'stringy' sense vanishes; therefore, $c_{\mathrm{str}}(D)$ is evaluated by (the 'identity manifestation' of)

$$\int_{\overline{D}} \mathbb{1}(0)\, d\mathfrak{c}_{\overline{D}} \quad .$$

The resolution $\overline{D}$ is not adequate to compute $c_{\mathrm{arc}}(D)$, since $\Omega^{\dim D}_{\overline{D}|D}$ is not invertible (that is, $\overline{D}$ does not 'resolve' the data in the sense of [3], §3.3). In order to obtain a resolution satisfying this condition, we blow up $\overline{D}$ further along $\overline{S}$: let $\widehat{D}$ be this blow-up, and denote by E the exceptional divisor; and let $\hat{\nu} : \widehat{D} \to D$ be the composition of the two blow-ups. Then the stringy relative canonical divisor is E, while $\Omega^{\dim D}_{\widehat{D}|D} \cong \mathcal{O}(2E)$.

This prompts us to propose the following (speculative) definition: for every nonnegative integer m, we can let $c^{(m)}(D)$ be the class in $(A_*D)_{\mathbb{Q}}$ corresponding to the celestial integral

$$\int_{\widehat{D}} \mathbb{1}(mE)\, d\mathfrak{c}_{\widehat{D}} \quad ,$$

so that $c^{(1)}(D) = c_{\mathrm{str}}(D)$ and $c^{(2)}(D) = c_{\mathrm{arc}}(D)$. As we will see in a moment, the class $c^{(m)}(D)$ has a well-defined 'limit as $m \to \infty$', which we will denote $c^{(\infty)}(D)$; the corresponding degrees will be denoted $\chi^{(m)}(D)$, $\chi^{(\infty)}(D)$.

Theorem 5.1. *With notation as in above and as in §3:*

$$2\,\varphi_* c(Y) = c^{(\infty)}(D) + 4\,c(O) \quad .$$

[10] This is called 'Ω flavor' in [3] and [5], as opposed to the (stringy) 'ω flavor'.

This statement is proved by computing $c^{(m)}$ explicitly; the following lemma does this, and gives a meaning to the limit of the class $c^{(m)}$ as $m \to \infty$.

Lemma 5.2. *With notation as above,*

$$c^{(m)}(D) = \pi_* c(\overline{D}) - c(S) + \frac{2}{1+m} c(S) \quad .$$

Proof. By definition of the integral (see [5], §7) we get

$$c^{(m)}(D) := \hat{\nu}_* \left(\frac{c(T\widehat{D})}{(1+E)} \left(1 + \frac{E}{1+m}\right) \cap [\widehat{D}] \right) = \hat{\nu}_* \left(c(T\widehat{D}) \cap [\widehat{D}] - \frac{m}{1+m_E} c(TE) \cap [E] \right) \tag{5.1}$$

$$= \hat{\nu}_* c_{\mathrm{SM}} \left(1\!\!1_{\widehat{D}} - \frac{m}{1+m} 1\!\!1_E \right) \quad . \tag{5.2}$$

Since $\overline{S}$ is nonsingular and of codimension 2 in $\overline{D}$, the exceptional divisor E is a $\mathbb{P}^1$-bundle over $\overline{S}$. As $\overline{S}$ maps isomorphically to S, we have

$$\hat{\nu}_*(1\!\!1_E) = 2 \cdot 1\!\!1_S \quad .$$

By the same token,

$$\hat{\nu}_*(1\!\!1_{\widehat{D}}) = \pi_*(1\!\!1_{\overline{D}}) + 1\!\!1_S \quad .$$

Thus

$$\hat{\nu}_* \left(1\!\!1_{\widehat{D}} - \frac{m}{1+m} 1\!\!1_E \right) = \pi_*(1\!\!1_{\overline{D}}) - 1\!\!1_S + \frac{2}{1+m} 1\!\!1_S \quad ,$$

and the statement follows by applying c_{SM} and using its properties (1), (2) listed in §4. □

The theorem follows immediately from Lemma 5.2 and Theorem 4.6.

Theorem 5.1 recovers the tadpole relation of §2.6 'without correction terms':

$$2\,\chi(Y) = \chi^{(\infty)}(D) + 4\,\chi(O) \quad ,$$

in full alignment with the string theory prediction obtained in §2.6. However, it is of course unclear at this point whether this is due to a lucky accident, or whether the formalism leading to the definition of $c^{(\infty)}(D)$ can really account for the relevant information at a good level of generality.

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